

FORCING CONSEQUENCES OF PFA TOGETHER WITH THE CONTINUUM LARGE

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ABSTRACT. We develop a new method for building forcing iterations with “symmetric systems of structures” as side conditions. Using our method we prove that the forcing axiom for the class of all the *finitely proper* posets of size ω_1 is compatible with $2^{\aleph_0} > \aleph_2$. In particular, this answers a question of Justin Moore by showing that \mathfrak{U} does not follow from this arithmetical assumption.

1. INTRODUCTION

Part of the progress in the study of forcing axioms includes the search for restricted forms of these axioms imposing limitations on the size of the real numbers. Given that forcing axioms typically imply $2^{\aleph_0} = \aleph_2$, a natural problem when faced with a consequence C of a forcing axiom is to find out whether C itself has any impact on the size of the continuum, and which. In this paper we approach this question by building a model in which a number of consequences of the forcing axiom for the class of all proper posets of size \aleph_1 (which we will call $\text{PFA}(\omega_1)$) hold and at the same time the continuum is arbitrarily large.

The standard strategy for producing models of Π_2 consequences (over the structure $\langle H(\omega_2), \in \rangle$) C of forcing axioms is by means of forcing iterations with the \aleph_2 -chain condition in which one keeps adding witnesses of the relevant Σ_1 facts. When the forcing axiom in question is PFA (or BPFA), the natural forcing for building a model of C is then the direct limit of a countable support iteration of proper forcings of length a suitable cardinal of cofinality greater than ω_1 and having the \aleph_2 -chain condition. By a theorem of Shelah, the limit of such an iteration is also a proper poset, and therefore ω_1 is preserved in the resulting generic extension. However, this type of construction will never give rise to a model in which $2^{\aleph_0} > \aleph_2$. The reason is the well-known

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general fact that a countable support iteration of non-trivial forcings of length λ , with $cf(\lambda) \geq \omega_1$, always adds generic filters for $Coll(\omega_1^V, \lambda)$ over all intermediate models.

In some cases, the question whether a given consequence of forcing axioms puts any upper bound on the size of the continuum can be settled by taking a model of the property in question together with the continuum small and adding many Cohen or random reals to it.

This way, it can be proved for example that the negation of *Club Guessing* does not imply $2^{\aleph_0} \leq \aleph_2$. Club Guessing (CG) asserts that there exists a ladder system $\langle A_\delta : \delta \in Lim(\omega_1) \rangle^1$ with the property that for every club $C \subseteq \omega_1$ there is some $\delta \in C$ such that a final segment of A_δ is included in C . It is not difficult to prove that the product with finite supports of Cohen forcing always preserves $\neg CG$.²

There are other anti-diamond principles for which the strategy of adding many Cohen reals does not work. This is for example the case of the negation of *Weak Club Guessing*. Weak Club Guessing (WCG) asserts that there exists a ladder system $\mathcal{A} = \langle A_\delta : \delta \in \Omega \rangle$ with the property that for every club $C \subseteq \omega_1$ there is some $\delta \in C$ such that $A_\delta \cap C$ is infinite. We will also say that \mathcal{A} is a WCG-sequence. Note that CG implies WCG. Also, using countable closed conditions it is not difficult to see that $\neg WCG$ is a consequence of BPFA, and that it can be forced over any model of CH by a countable support iteration of proper forcings of length ω_2 . On the other hand, Cohen forcing always adds a WCG-sequence.³ Hence, the consistency of $\neg WCG$ with the continuum of size strictly larger than \aleph_2 cannot be proved by adding Cohen reals to a model where WCG is false. One possibility for this is to add many random reals to a model of $\neg WCG$. In fact, it is not hard to see that random forcing always preserves $\neg WCG$.⁴

There are however some strengthenings of $\neg CG$ for which the above methods do not apply. For example this is the case of *Code(even-odd)*, a principle formulated by Tadatashi Miyamoto saying that WCG fails in a dramatic way. More precisely, *Code(even-odd)* says that for every ladder system $\mathcal{A} = \langle A_\delta : \delta \in Lim(\omega_1) \rangle$ and every $B \subseteq \omega_1$ there exist two clubs C and D of ω_1 such that for each $\delta \in C$: If $\delta \in B$ (resp. $\delta \notin B$), then $|A_\delta \cap D| < \aleph_0$ is odd (resp. even).

Another example is the negation of \mathfrak{U} (mho). The principle \mathfrak{U} , formulated by Moore, says that there is a sequence $\langle f_\alpha : \alpha \in \omega_1 \rangle$ such

¹A sequence $\langle A_\delta : \delta \in Lim(\omega_1) \rangle$ is a *ladder system* iff for every $\delta \in Lim(\omega_1)$, A_δ is a cofinal subset of δ of order type ω .

²We learned this from Hiroshi Sakai.

³This has been observed as well by Sakai.

⁴We learned this from Michael Hrusak.

that f_α is a continuous map⁵ from α into ω for all $\alpha \in \omega_1$ and with the property that for every club $E \subseteq \omega_1$ there is a δ in E such that f_δ takes all values in ω on $E \cap \delta$. The following is an observations of Moore concerning this statement: Notice that if $\alpha < \omega_1$ and $f : \alpha \rightarrow \omega$ is continuous, then α can be partitioned into open intervals on which f is constant. In such a situation there is a cofinal $C \subseteq \alpha$ of order-type at most ω such that $f(\varepsilon)$ depends only on the size of $\varepsilon \cap C$. From this it is clear that \mathfrak{U} follows from CG. In [6] Moore asked whether \mathfrak{U} follows from $2^{\aleph_0} > \aleph_2$.

In this paper we introduce an alternative method to standard countable support iterations for producing models of certain Π_2 statements. Using this method we prove that a certain forcing axiom which is a natural fragment of $\text{PFA}(\omega_1)$ and which implies both $\text{Code}(\text{even-odd})$ and the negation of \mathfrak{U} is consistent together with the continuum being larger than \aleph_2 . In fact, we build a generic extension where this forcing axiom holds and 2^{\aleph_0} is equal to κ , where κ is an arbitrarily fixed cardinal satisfying certain GCH like assumptions in the ground model.

Definition 1.1. Given a poset P , we will say that P is *finitely proper* iff for every regular cardinal $\lambda \geq \aleph_2$, every finite set $\{N_i : i \in m\}$ of countable elementary substructures of $H(\lambda)$ containing P and every condition $p \in \bigcap \{N_i : i < m\} \cap P$ there is a condition p' extending p and (N_i, P) -generic for all i .

Definition 1.2. Let $\text{PFA}^*(\omega_1)$ denote $\text{FA}(\Gamma)$, where Γ is the class of all the finitely proper posets of size \aleph_1 .

Note that all partial orders satisfying the countable chain condition are finitely proper, and therefore $\text{PFA}^*(\omega_1)$ is a generalization of MA_{ω_1} . Other applications of $\text{PFA}^*(\omega_1)$ (including the failure of \mathfrak{U} and some other anti-diamond principles on ω_1) can be found in section 4.

Our main theorem is the following.

Theorem 1.3. (CH) *If κ is a cardinal such that $\kappa^{\aleph_1} = \kappa$ and $2^{<\kappa} = \kappa$, then there exists a proper forcing notion \mathcal{P} with the \aleph_2 -chain condition such that both $\text{PFA}^*(\omega_1)$ and $2^{\aleph_0} = \kappa$ hold in the generic extension by \mathcal{P} .*

Our method produces a proper forcing notion with the \aleph_2 -chain condition. This forcing notion \mathcal{P} is the direct limit \mathcal{P}_κ of a sequence $\langle \mathcal{P}_\alpha : \alpha < \kappa \rangle$ of partial orders, where \mathcal{P}_α is a complete suborder of \mathcal{P}_β whenever α is less than β . Our construction can thus be seen as a forcing iteration in a broad sense.

⁵With respect to the order topology.

One crucial feature in the proof of properness is the use of certain finite “symmetric systems” of side conditions. These side conditions will be elementary substructures of $H(\kappa)$ and will be added by \mathcal{P}_0 . If N is one of them, $q = (p, \Delta) \in \mathcal{P}_\alpha$ and $(N, \alpha) \in \Delta$, then all the relevant pieces of information coming from any \mathcal{P}_α -extension of q can be relativized to N . This happens because – under the above assumptions – if $p(\xi) \neq 0$ and $\xi \in \alpha \cap N$, then $p(\xi)$ is asked to be generic over $N[G_\xi]$ with respect to a certain (finitely) poset whose domain is included in ω_1 . Each $p(\xi)$ will be a countable ordinal and the set of those $\xi < \alpha$ on which $p(\xi)$ is nonzero is finite. The technique of ensuring properness of a given forcing notion by incorporating elementary substructures of some large enough model into its definition may be traced back to Todorćević’s [7].

The rest of the paper is organized as follows: Section 2 starts with a definition of a partial order \mathcal{P} witnessing Theorem 1.3. Section 3 contains proofs of the main facts about \mathcal{P} . Theorem 1.3 follows then easily from these facts. Finally, Section 4 deals with some applications of $\text{PFA}^*(\omega_1)$.

Even if this work tries to be sufficiently self-contained, we will assume that the reader has a good knowledge of forcing. Two good references are Kunen ([3]) and Jech ([2]). Our notation is more or less standard, but in some cases we have tried to give a complete explanation of the symbols and notions that we use.

2. THE FORCING CONSTRUCTION

The proof of Theorem 1.3 will be given in a sequence of lemmas.

Let $\Phi : \kappa \rightarrow H(\kappa)$ be a surjection such that for every x in $H(\kappa)$, $\Phi^{-1}(\{x\})$ is unbounded in κ .⁶

Let $\langle \theta_\alpha : \alpha \leq \kappa \rangle$ be the strictly increasing sequence of regular cardinals defined as $\theta_0 = |2^\kappa|^+$ and $\theta_\alpha = |2^{\sup\{\theta_\beta : \beta \leq \alpha\}}|^+$ if $\alpha > 0$. For each $\alpha \leq \kappa$ let \mathcal{M}_α^* be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing Φ and $\langle \theta_\beta : \beta < \alpha \rangle$. Let also $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) : N^* \in \mathcal{M}_\alpha^*\}$ and note that if $\alpha < \beta$, then \mathcal{M}_α^* belongs to all members of \mathcal{M}_β^* containing the ordinal α .

Our forcing \mathcal{P} will be the direct limit \mathcal{P}_κ of a certain sequence $\langle \mathcal{P}_\alpha : \alpha < \kappa \rangle$ of forcings. The properness of each \mathcal{P}_α will be witnessed by the club \mathcal{M}_α^* . The main idea here is to use the elements of \mathcal{M}_α as side conditions to ensure properness, but without losing the \aleph_2 -chain condition. This brings us to the idea of *symmetric systems*.

⁶The bookkeeping function Φ exists by the assumption $2^{<\kappa} = \kappa$.

Definition 2.1. If N is such that $N \cap \omega_1$ is an ordinal, then δ_N will denote this ordinal.

Definition 2.2. Let $\bar{\kappa}$ be an uncountable cardinal, let \mathcal{M} be club of $[H(\bar{\kappa})]^{\aleph_0}$, let $P \subseteq H(\bar{\kappa})$, and let $\{N_i : i < m\}$ be a finite set of members of \mathcal{M} . We will say that $\{N_i : i < m\}$ is a *P-symmetric system of members of \mathcal{M}* if

- (A) For every $i < m$, $N_i \in \mathcal{M}$.
- (B) Given distinct i, i' in m , if $\delta_{N_i} = \delta_{N_{i'}}$, then there is a (unique) isomorphism

$$\Psi_{N_i, N_{i'}} : (N_i, \in, P \cap N_i) \longrightarrow (N_{i'}, \in, P \cap N_{i'})$$

(Note that $\Psi_{N_i, N_{i'}}$ is continuous, in the sense that $\sup(\Psi_{N_i, N_{i'}} \text{“}\xi\text{”}) = \Psi_{N_i, N_{i'}}(\xi)$ whenever $\xi \in N_i$ is an ordinal of countable cofinality.)⁷

Furthermore, we ask that $\Psi_{N_i, N_{i'}}$ be the identity on $\kappa \cap N_i \cap N_{i'}$.

- (C) For all i, j in m , if $\delta_{N_j} < \delta_{N_i}$, then there is some $i' < m$ such that $\delta_{N_{i'}} = \delta_{N_i}$ and $N_j \in N_{i'}$.
- (D) For all i, i', j in m , if $N_j \in N_i$ and $\delta_{N_i} = \delta_{N_{i'}}$, then there is some $j' < m$ such that $\Psi_{N_i, N_{i'}}(N_j) = N_{j'}$.

If \mathcal{M} is the club of countable $N \preceq H(\bar{\kappa})$ and $P = \emptyset$, we will call $\{N_i : i < m\}$ a *symmetric system of elementary substructures of $H(\bar{\kappa})$* .

Let us proceed to the definition of $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ now.

\mathcal{P}_0 consist of all pairs of the form $(\emptyset, \{(N_i, 0) : i < m\})$, where $\{N_i : i < m\}$ is a Φ -symmetric system of members of \mathcal{M}_0 .

Given \mathcal{P}_0 -conditions $q_\epsilon = (\emptyset, \{(N_i^\epsilon, 0) : i < m_\epsilon\})$ for $\epsilon \in \{0, 1\}$, q_1 extends q_0 if $\{N_i^0 : i < m_0\} \subseteq \{N_i^1 : i < m_1\}$.

In the definition of a \mathcal{P}_0 -condition we have used the empty set in a completely vacuous way. These (vacuous) \emptyset 's are there to ensure that the (uniformly defined) operation of restricting a condition in a (further) \mathcal{P}_α to an ordinal $\beta < \alpha$ yields a condition in \mathcal{P}_β when applied to any condition in any \mathcal{P}_α and to $\beta = 0$.

Let $\alpha < \kappa$ and suppose that we have defined \mathcal{P}_α and that \mathcal{P}_α is a partial order of size κ with the \aleph_2 -chain condition.⁸ From now on, we

⁷Equivalently, $\Psi_{N_i, N_{i'}}$ is an isomorphism between the structures $(N_i, \in, N_i \cap cf(\omega))$ and $(N_{i'}, \in, N_{i'} \cap cf(\omega))$.

⁸That \mathcal{P}_α is a partial order will follow easily from the definitions we are about to see. $|\mathcal{P}_\alpha| = \kappa$ follows from $\kappa > \omega$ and $2^{<\kappa} = \kappa$. The \aleph_2 -chain condition of \mathcal{P}_α follows from Lemma 3.2.

also assume that if $q \in \mathcal{P}_\alpha$, then q is an ordered pair of the form (p, Δ) , where

- (\circ) p is a sequence of ordinals of length α and
- (\circ) Δ is a finite relation $\{(N_i, \tau_i) : i \in n\}$ whose domain and range are included in, respectively, \mathcal{M}_0 and $\alpha + 1$.

Definition 2.3. If $q = (p, \Delta)$ is an element of \mathcal{P}_α , then

- (\bullet) \mathcal{X}_q denotes the domain of Δ and
- (\bullet) \mathcal{X}_q^* denotes the set of those N such that $(N, \alpha) \in \Delta$.

Definition 2.4. (In $V[G_\alpha]$, where G_α is a \mathcal{P}_α -generic filter) Let \mathcal{R} be a forcing on ω_1 whose weakest condition is the ordinal 0. We will say that \mathcal{R} is *V-finitely proper (with respect to $V^{\mathcal{P}_\alpha}$)* iff there exists a club $D \subseteq ([H(\kappa)]^{\aleph_0})^V$ in V such that:

If $\{N_i : i \in m\} \subseteq D$ ($m \in \omega$) is such that $\{N_i : i < m\} \subseteq \mathcal{X}_u^*$ for some $u \in G_\alpha$ and such that $\mathcal{R} \in N_i[G_\alpha]$ for all i , then for every $\nu \in \bigcap \{N_i \cap \omega_1 : i \in m\}$ there exists ν^* such that ν^* \mathcal{R} -extends ν and is $(N_i[G_\alpha], \mathcal{R})$ -generic for all i .

The definition of $\mathcal{P}_{\alpha+1}$ is as follows. Conditions in $\mathcal{P}_{\alpha+1}$ are pairs of the form

$$q = (p^\frown \langle \nu \rangle, \{(N_i, \beta_i) : i < m\})$$

with the following properties.

- (b0) p is a sequence of length α .⁹
- (b1) For all $i < m$, $\beta_i \leq (\alpha + 1) \cap \sup(N_i \cap \kappa)$.
- (b2) The restriction of q to α is a condition in \mathcal{P}_α . This restriction is defined as the object

$$q|_\alpha := (p, \{(N_i, \beta_i^\alpha) : i < m\}),$$

where $\beta_i^\alpha = \min\{\beta_i, \alpha\}$.

- (b3) $\nu \in \omega_1$
- (b4) If $\Phi(\alpha) = \dot{\mathcal{R}}$ is a \mathcal{P}_α -name for a V -finitely proper forcing on ω_1 whose weakest condition is 0, $\nu \neq 0$, $N \in \mathcal{X}_q^*$, and $N \in \mathcal{M}_{\alpha+1}$, then $q|_\alpha$ forces (in \mathcal{P}_α) that ν is $(N[\dot{G}_\alpha], \dot{\mathcal{R}})$ -generic.
- (b5) If $\Phi(\alpha)$ is not relevant (i.e., if $\Phi(\alpha)$ is not a \mathcal{P}_α -name for a V -finitely proper forcing on ω_1 whose weakest condition is 0), then $\nu = 0$.

Given conditions

$$q_\epsilon = (p_\epsilon^\frown \langle \nu^\epsilon \rangle, \{(N_i^\epsilon, \beta_i^\epsilon) : i < m_\epsilon\})$$

⁹Note that $p^\frown \langle \nu \rangle$ is a sequence of length $\alpha + 1$.

(for $\epsilon \in \{0, 1\}$) in $\mathcal{P}_{\alpha+1}$, we will say that $q_1 \leq_{\alpha+1} q_0$ if and only if the following holds.

- (c 1) $q_1|_{\alpha} \leq_{\alpha} q_0|_{\alpha}$.
- (c 2) If $\Phi(\alpha) = \dot{\mathcal{R}}$ is a \mathcal{P}_{α} -name for a V -finitely proper forcing on ω_1 whose weakest condition is 0, then $q_1|_{\alpha}$ forces in \mathcal{P}_{α} that $\nu^1 \dot{\mathcal{R}}$ -extends ν^0 .
- (c 3) For all $i < m_0$ there exists $\tilde{\beta}_i \geq \beta_i^0$ such that $(N_i^0, \tilde{\beta}_i) \in \Delta_{q_1}$.

Some explanations concerning the natural embedding between \mathcal{P}_{α} and $\mathcal{P}_{\alpha+1}$ can be found in the last paragraph of this section and in lemma 3.1. In Remark 3.5 we also show that if $\Phi(\alpha)$ is relevant and $q = (p \smallfrown \langle \nu \rangle, \Delta_q)$ is an element of $\mathcal{P}_{\alpha+1}$, then we can find a condition $q' = (p' \smallfrown \langle \nu' \rangle, \Delta'_q)$ $\mathcal{P}_{\alpha+1}$ -extending q and such that $\nu' \neq 0$.

Now assume that $\gamma \leq \kappa$ is a nonzero limit ordinal. A pair of the form $q = (p, \{(N_i, \beta_i) : i < m\})$ is a condition in \mathcal{P}_{γ} if and only if it has following properties.

- (d 1) p is a sequence of length γ .
- (d 2) For all $i < m$, $\beta_i \leq \gamma \cap \sup(N_i \cap \kappa)$.¹⁰
- (d 3) For every $\varepsilon < \gamma$, the restriction $q|_{\varepsilon} := (p \upharpoonright \varepsilon, \{(N_i, \beta_i^{\varepsilon}) : i < m\})$ is a condition in $\mathcal{P}_{\varepsilon}$; where $\beta_i^{\varepsilon} = \beta_i$ if $\beta_i \leq \varepsilon$, and $\beta_i^{\varepsilon} = \varepsilon$ if $\beta_i > \varepsilon$.
- (d 4) The set of ordinals $\zeta < \gamma$ such that $p(\zeta) \neq 0$ is finite.

Finally, given $q_1 = (p_1, \Delta_1)$ and $q_0 = (p_0, \Delta_0)$ in \mathcal{P}_{γ} , we will say that $q_1 \leq_{\gamma} q_0$ if and only if the following holds.

- (e) For every $\beta < \gamma$, $q_1|_{\beta} \leq_{\beta} q_0|_{\beta}$.¹¹

Definition 2.5. Given any α and any \mathcal{P}_{α} -condition $q = (p, \{(N_i, \beta_i) : i < n\})$, the finite set of $\zeta < \alpha$ such that $p(\zeta) \neq 0$ will be denoted by $\text{supp}(q)$ and will be called the *support* of q .

If $\alpha < \beta \leq \kappa$ and $q = (p, \{(N_j, \beta_j) : j < m\})$ is a \mathcal{P}_{β} -condition such that $\text{supp}(q) \subseteq \alpha$ and $\beta_j \leq \alpha$ for all j , we will often identify q with the \mathcal{P}_{α} -condition $(p \upharpoonright \alpha, \{(N_j, \beta_j) : j < m\})$. Notice that if $\alpha < \beta \leq \kappa$ and $q = (p, \Delta)$ is a \mathcal{P}_{α} -condition, then q can be naturally identified with a \mathcal{P}_{β} -condition (p^*, Δ) by setting $p^* \upharpoonright \alpha = p$ and letting $p^*(\xi) = (\emptyset, \emptyset)$ for all $\xi \in [\alpha, \beta)$.¹² Therefore if q is a \mathcal{P}_{β} -condition and $\alpha < \beta$, then q \mathcal{P}_{β} -extends its restriction to α . Finally notice that if $\alpha < \beta \leq \kappa$, $N^* \in \mathcal{M}_{\beta}^*$ and $\alpha \in N^*$, then $\mathcal{P}_{\alpha} \in N^*$.

¹⁰Note that β_i is always less than κ (even when $\gamma = \kappa$).

¹¹Notice that $(p_1, \Delta_1) \leq_{\gamma} (p_0, \Delta_0)$ implies that for every $(N_i, \beta_i) \in \Delta_0$ there exists $\tilde{\beta}_i \geq \beta_i$ such that $(N_i, \tilde{\beta}_i) \in \Delta_1$.

¹²This is in accordance with the above identification.

3. THE MAIN FACTS

We are going to prove the relevant properties of the forcings \mathcal{P}_α . Theorem 1.3 will follow immediately from them.

Lemma 3.1. *Let $\alpha \leq \beta \leq \kappa$. If $q = (p, \Delta_q) \in \mathcal{P}_\alpha$, $s = (r, \Delta_s) \in \mathcal{P}_\beta$ and $q \leq_\alpha s|_\alpha$, then $(p^\frown(r \upharpoonright [\alpha, \beta)), \Delta_q \cup \Delta_s)$ is a condition in \mathcal{P}_β extending s . Therefore, any maximal antichain in \mathcal{P}_α can be seen as a maximal antichain in \mathcal{P}_β and \mathcal{P}_α can be seen as a complete suborder of \mathcal{P}_β .*

Proof. This is a mechanical verification. One crucial point is the use of the markers β_i in the definition of the forcing. New side conditions (N_i, β_i) appearing in Δ_q may well have the property that $N_i \cap [\alpha, \beta) \neq \emptyset$, but they will not impose any problematic promises – coming from clause (b4) in the successor case of the definition – on ordinals ν occurring in $r \upharpoonright [\alpha, \beta)$. The reason is simply that $\beta_i \leq \alpha$. \square

The next step in the proof of Theorem 1.3 will be to show that all \mathcal{P}_α (for $\alpha \leq \kappa$) have the \aleph_2 –chain condition. We will actually show that these forcings are \aleph_2 –Knaster.¹³

Lemma 3.2. (CH) *For every ordinal $\alpha \leq \kappa$, \mathcal{P}_α is \aleph_2 –Knaster.*

Proof. The proof is by induction on α and involves standard Δ –system and pigeonhole principle arguments. The conclusion for $\alpha = 0$ follows from CH: Suppose $m < \omega$ and $q_\xi = \{N_i^\xi : i < m\}$ is a \mathcal{P}_0 –condition for each $\xi < \omega_2$.¹⁴ By CH we may assume that $\{\bigcup_{i < m} (N_i^\xi \cap \kappa) : \xi < \omega_2\}$ forms a Δ –system with root X . Furthermore, by CH we may assume, for all $\xi, \xi' < \omega_2$ and $i, i' < m$ such that $\delta_{N_i^\xi} = \delta_{N_{i'}^{\xi'}}$, that the structures $\langle N_i^\xi, \in, X \cap N_i^\xi \rangle$ and $\langle N_{i'}^{\xi'}, \in, X \cap N_{i'}^{\xi'} \rangle$ are isomorphic, and that the corresponding isomorphism fixes $N_i^\xi \cap N_{i'}^{\xi'}$. Now it is easy to check, for all $\xi < \xi'$, that $q_\xi \cup q_{\xi'}$ extends both q_ξ and $q_{\xi'}$.

For $\alpha = \sigma + 1$ suppose $q_\xi = (p_\xi^\frown \langle \nu_\xi \rangle, \Delta_\xi)$ is a $\mathcal{P}_{\sigma+1}$ –condition for each $\xi < \omega_2$. We may assume by our induction hypothesis that all $q_\xi|_\sigma$ are pairwise compatible. Let $r_{\xi, \xi'} \in \mathcal{P}_\sigma$ be a condition extending both $q_\xi|_\sigma$ and $q_{\xi'}|_\sigma$ for all $\xi < \xi' < \omega_2$. Now find a set $I \subseteq \omega_2$ of size \aleph_2 such that for all $\xi < \xi'$ in I , $\nu_\xi = \nu_{\xi'}$. It is clear now, for all such ξ, ξ' , that the natural amalgamation of $r_{\xi, \xi'}$, q_ξ and $q_{\xi'}$ is a $\mathcal{P}_{\sigma+1}$ –condition extending q_ξ and $q_{\xi'}$.

¹³A forcing P is μ –Knaster if every subset of P of cardinality μ includes a subset of cardinality μ of pairwise compatible conditions.

¹⁴We are identifying a \mathcal{P}_0 –condition q with \mathcal{X}_q^* , which is fine for this proof.

For α a nonzero limit ordinal, suppose q_ξ is a \mathcal{P}_α -condition for all $\xi < \omega_2$. Suppose $cf(\alpha) < \omega_2$. There is $\sigma < \alpha$ such that $I = \{\xi < \omega_2 : \text{supp}(q_\xi) \subseteq \sigma\}$ has size \aleph_2 . By induction hypothesis there is $I' \subseteq I$ of size \aleph_2 such that all $q_\xi \restriction \sigma$ (for $\xi \in I'$) are pairwise compatible in \mathcal{P}_σ . But now it is easy to verify for all $\xi < \xi'$ in I that q_ξ and $q_{\xi'}$ are compatible in \mathcal{P}_α .

Finally, suppose $cf(\alpha) \geq \omega_2$. For each $\xi < \omega_2$, let Z_ξ be equal to the union of the sets $\text{supp}(q_\xi)$ and $\kappa \cap \bigcup \mathcal{X}_{q_\xi}$. By CH we may find $I \subseteq \omega_2$ of size \aleph_2 such that $\{Z_\xi : \xi \in I\}$ forms a Δ -system with root X (Note that $\{\text{supp}(q_\xi) : \xi \in I\}$ forms a Δ -system too).

Let now $\sigma < \alpha$ be such that $X \subseteq \sigma$ (σ exists by $cf(\alpha) \geq \omega_1$). By induction hypothesis we may assume that all $q_\xi \restriction \sigma$ are pairwise compatible in \mathcal{P}_σ . For all $\xi < \xi'$ in I let $r_{\xi, \xi'}$ be a condition in \mathcal{P}_σ extending $q_\xi \restriction \sigma$ and $q_{\xi'} \restriction \sigma$. It is not difficult to check, for all such ξ, ξ' , that the natural amalgamation of $r_{\xi, \xi'}$, q_ξ and $q_{\xi'}$ is a \mathcal{P}_α -condition extending q_ξ and $q_{\xi'}$. \square

Definition 3.3. Given $\alpha \leq \kappa$, a condition $q \in \mathcal{P}_\alpha$, and a countable elementary substructure $N \prec H(\kappa)$, we will say that q is (N, \mathcal{P}_α) -pre-generic in case

- (o) $\alpha < \kappa$ and the pair (N, α) is in Δ_q , or else
- (o) $\alpha = \kappa$ and the pair $(N, \text{sup}(N \cap \kappa))$ is in Δ_q .

The properness of all \mathcal{P}_α is an immediate consequence of the following lemma.

Lemma 3.4. Suppose $\alpha \leq \kappa$ and $N^* \in \mathcal{M}_\alpha^*$. Let $N = N^* \cap H(\kappa)$. Then the following conditions hold.

- (1) $_\alpha$ For every $q \in N$ there is $q' \leq_\alpha q$ such that q' is (N, \mathcal{P}_α) -pre-generic.
- (2) $_\alpha$ If $\mathcal{P}_\alpha \in N^*$ and $q \in \mathcal{P}_\alpha$ is (N, \mathcal{P}_α) -pre-generic, then q is $(N^*, \mathcal{P}_\alpha)$ -generic.

Proof. The proof will be by induction on α . We start with the case $\alpha = 0$. For simplicity we are going to identify a \mathcal{P}_0 -condition q with \mathcal{X}_q . The proof of (1) $_0$ is trivial: It suffices to set $q' = q \cup \{(N, 0)\}$.

The proof of (2) $_0$ is also easy: Let E be a dense subset of \mathcal{P}_0 in N^* . It suffices to show that there is some condition in $E \cap N^*$ compatible with q . Notice that $q \cap N^* \in \mathcal{P}_0$. Hence, we may find a condition $q^\circ \in E \cap N^*$ extending $q \cap N^*$. Now let

$$q^* = q \cup \{\Psi_{N, \overline{N}}(M) : M \in q^\circ, \overline{N} \in \mathcal{X}_q, \delta_{\overline{N}} = \delta_N\}$$

It takes a routine verification to check that q^* is a condition in \mathcal{P}_0 extending both q and q° .

Let us proceed to the more substantial case $\alpha = \sigma + 1$. We start by proving $(1)_\alpha$. Let $q = (p^\frown \langle \nu \rangle, \Delta_q)$. By $(1)_\sigma$ we can assume that there is a condition $t \in \mathcal{P}_\sigma$ extending $q|_\sigma$ and \mathcal{P}_σ -pre-generic for N . Without loss of generality we can assume that $\Phi(\sigma) = \dot{\mathcal{R}}$ is a \mathcal{P}_σ -name for a V -finitely proper forcing on ω_1 with weakest condition 0, and that $\nu \neq 0$. In $V^{\mathcal{P}_\sigma|t}$ let \dot{D} be a club in $V \cap N^*[\dot{G}_\sigma]$ witnessing the V -finite properness of $\dot{\mathcal{R}}$ and note that $N \in \dot{D}$ since $N^*[\dot{G}_\sigma] \cap V = N^*$ (which follows from $(2)_\sigma$ and from the fact that t is \mathcal{P}_σ -pre-generic for N). By the definition of V -finite properness (in fact we are using something weaker which may be called V -properness¹⁵) and the fact that $N \in \mathcal{X}_t^*$, there is then some $z = (y, \Delta_z) \in \mathcal{P}_\sigma$ extending t and an ordinal ν^* such z forces that ν^* is an $(N[\dot{G}_\sigma], \dot{\mathcal{R}})$ -generic condition $\dot{\mathcal{R}}$ -extending ν . It suffices to define q' as the condition $(y^\frown \langle \nu^* \rangle, \Delta_q \cup \Delta_z \cup \{(N, \alpha)\})$.

Remark 3.5. Starting from $\nu = 0$, we can also run the same argument and find a condition $q' = (p'^\frown \langle \nu' \rangle, \Delta'_q)$ extending q and such that $\nu' \neq 0$.

Proof. This is true since $(2)_\sigma$ guarantees that $q|_\sigma$ is also $(M^*, \mathcal{P}_\sigma)$ -generic for all $M \in \mathcal{X}_q^*$ (which implies that the above t forces that all these M are in \dot{D}). The only difference is that we use here the stronger assumption of V -finite properness (and not only of V -properness). \square

Now let us prove $(2)_\alpha$. Again, we may assume that $\Phi(\sigma) = \dot{\mathcal{R}}$ is a \mathcal{P}_σ -name for a V -finitely proper forcing on ω_1 with weakest condition 0 (the proof in the other case is easier). Let E be an open dense set of \mathcal{P}_α in N^* . We should find a condition $\tilde{q} \in E \cap N^*$ compatible with q . Since E is open, we may start assuming that $q \in E$. Let G_σ be a \mathcal{P}_σ -generic filter over V with $q|_\sigma \in G_\sigma$. Using $(2)_\sigma$ it must be clear that G_σ is also generic over N^* . Define E/G_σ as the set of those conditions in E whose restriction to σ belongs to G_σ , and \tilde{E} as the set of those countable ordinals η such that either

- (i) there exists some $t \in E/G_\sigma$ of the form $t = (s^\frown \langle \eta \rangle, \Delta_t)$, or else
- (ii) there is no η' $\Phi(\sigma)$ -extending η for which there is any $t \in E/G_\sigma$ of the form $t = (s^\frown \langle \eta' \rangle, \Delta_t)$

Note that \tilde{E} is a dense subset of $\dot{\mathcal{R}}^{G_\sigma}$ and that $\tilde{E} \in N^*[G_\sigma]$. Assume now that q is of the form $q = (p^\frown \langle \nu \rangle, \Delta_q)$. If $\nu \neq 0$, then by condition (b4) in the definition of \mathcal{P}_α we know that there exists $\eta \in \tilde{E} \cap N^*[G_\sigma] \cap \omega_1$ such that ν and η are $\dot{\mathcal{R}}^{G_\sigma}$ -compatible.

Claim. *Condition (i) above holds for η .*

¹⁵This would be the variant of Definition 2.4 in which finite sets of structures are replaced by individual structures.

Proof. Let $r = (s, \Delta_r)$ be a condition in G extending $q|_\sigma$ and let η' such that r forces that η' is a condition in $\Phi(\sigma)$ extending both η and ν . But then $q^* := (s^\wedge \langle \eta' \rangle, \Delta_r \cup \Delta_q)$ is a \mathcal{P}_α condition extending q , $q^* \in E/G_\sigma$ and $q^*|_\sigma$ forces that condition (i) holds for η . This shows that $q|_\sigma$ forces that condition (i) holds for η . \square

By the above claim and by $N^*[G_\sigma] \prec H(\theta_\sigma)^V[G_\sigma]$, there is $t = (s^\wedge \langle \eta \rangle, \Delta_t)$ in $E/G_\sigma \cap N^*[G_\sigma]$, and of course $t \in N$ since $N^*[G_\sigma] \cap V = N^*$ by $(2)_\sigma$. It remains to see that $\tilde{q} = t$ is compatible with q . For this, notice that there is some $w = (w_0, \Delta_w) \in G_\sigma$ extending $q|_\sigma$ and $t|_\sigma$ and there is some $\eta^* < \omega_1$ such that w forces that η^* $\dot{\mathcal{R}}^{G_\sigma}$ -extends η and ν . But then w forces that η^* is $(M^*[G_\sigma], \dot{\mathcal{R}}^{G_\sigma})$ -generic whenever $M^* \in \mathcal{M}_\alpha^*$ is such that $(M^* \cap H(\kappa), \alpha) \in \Delta_t \cup \Delta_q$ since η^* extends η and ν . It follows that $(w_0^\wedge \langle \eta^* \rangle, \Delta_q \cup \Delta_t \cup \Delta_w)$ is a common extension of q and t . If $\nu = 0$, then $q = (p^\wedge \langle 0 \rangle, \Delta_q) \in E/G_\sigma$. Using $(2)_\sigma$ again, we know that there must be a condition $t = (s^\wedge \langle 0 \rangle, \Delta_r) \in E/G_\sigma \cap N^*$. Just as before, it suffices to let $\tilde{q} = t$.

It remains to prove the lemma for the case when α is a nonzero limit ordinal. The proof of $(1)_\alpha$ is trivial using $(1)_\beta$ for all $\beta < \alpha$, together with the fact the support of any \mathcal{P}_α -condition is bounded in α . For $(2)_\alpha$, let $E \subseteq \mathcal{P}_\alpha$ be dense and open, $E \in N^*$, and let q satisfy the hypothesis of $(2)_\alpha$. We want to find a condition in $E \cap N^*$ compatible with q . We may assume that $q \in E$.

Suppose first that $cf(\alpha) = \omega$. In this case we may take $\sigma \in N^* \cap \alpha$ above $\text{supp}(q)$. Let G_σ be \mathcal{P}_σ -generic with $q|_\sigma \in G_\sigma$. In $N^*[G_\sigma]$ it is true that there is a condition $q^\circ \in \mathcal{P}_\alpha$ such that

- (a) $q^\circ \in E$ and $q^\circ|_\sigma \in G_\sigma$, and
- (b) $\text{supp}(q^\circ) \subseteq \sigma$.

(the existence of such a q° is witnessed in $V[G_\sigma]$ by q .)

Since $q|_\sigma$ is $(N^*, \mathcal{P}_\sigma)$ -generic by induction hypothesis, $q^\circ \in N^*$. By extending q below σ if necessary, we may assume that $q|_\sigma$ decides q° and extends $q^\circ|_\sigma$. But now, if $q = (p, \Delta_q)$, the natural amalgamation $(p, \Delta_q \cup \Delta_{q^\circ})$ of q and q° is a \mathcal{P}_α -condition extending them.

Finally, suppose $cf(\alpha) \geq \omega_1$. Notice that if $N' \in \mathcal{X}_q$ and $\delta_{N'} < \delta_N$, then $\text{sup}(N' \cap N \cap \alpha) \leq \text{sup}(\Psi_{\overline{N}, N}(N') \cap \alpha) \in N \cap \alpha$ ¹⁶ whenever $\overline{N} \in \mathcal{X}_q$ is such that $\delta_{\overline{N}} = \delta_N$ and $N' \in \overline{N}$. Hence we may fix $\sigma \in N \cap \alpha$ above $\text{supp}(q) \cap N$ and above $\text{sup}(N' \cap N \cap \alpha)$ for all $N' \in \mathcal{X}_q$ with $\delta_{N'} < \delta_N$.

As in the above case, if G_σ is \mathcal{P}_σ -generic with $q|_\sigma \in G_\sigma$, then in $N^*[G_\sigma]$ we can find a condition $q^\circ = (p^\circ, \Delta_\circ) \in \mathcal{P}_\alpha$ such that $q^\circ \in E$

¹⁶Recall that $\Psi_{\overline{N}, N}$ fixes $\overline{N} \cap N \cap \kappa$.

and $q^\circ|_\sigma \in G_\sigma$ (again, the existence of such a condition is witnessed in $V[G_\sigma]$ by q), and such a q° will necessarily be in N^* . By extending q below σ we may assume that $q|_\sigma$ decides q° and extends $q^\circ|_\sigma$. The proof of $(2)_\alpha$ in this case will be finished if we can show that there is a condition q^\dagger extending q and q° . The condition q^\dagger can be built by recursion on $\text{supp}(q) \cup \text{supp}(q^\circ)$. This finite construction mimics the proof of $(1)_\beta$ for successor β , but also uses the strong assumption of V -finite properness. Note for instance that if η is in the support of q° and $\sigma \leq \eta < \alpha$, then $p^\circ(\eta) = \pi$ is an ordinal in the intersection of all N' with $N' \in \mathcal{X}_q$, $\eta \in N'$ and $\delta_{N'} \geq \delta_{N^*}$. Hence, if $\Phi(\eta) = \dot{\mathcal{R}}$ is a \mathcal{P}_η -name for a V -finitely proper poset on ω_1 with 0 as weakest condition, then there is an ordinal π^* and a common extension of $q|_\eta$ and $q^\circ|_\eta$ forcing that π^* $\dot{\mathcal{R}}$ -extends π and is $(N^*[\dot{G}_\eta], \dot{\mathcal{R}})$ -generic for all relevant N^* . This finishes the proof of $(2)_\alpha$ for limit α and the proof of the lemma. \square

Corollary 3.6. *For every $\alpha \leq \kappa$, \mathcal{P}_α is proper.*

Given an ordinal $\alpha < \kappa$, we let \dot{G}_α^+ be a $\mathcal{P}_{\alpha+1}$ -name for the collection of all ν for which there exists a condition $q \in \dot{G}_{\alpha+1}$ of the form $q = (p^\frown \langle \nu \rangle, \{(N_k, \beta_k) : k < m\})$.

The following lemmas are easy.

Lemma 3.7. *If $\alpha < \kappa$ and $\Phi(\alpha) = \dot{\mathcal{R}}$ is a \mathcal{P}_α -name for a V -finitely proper forcing on ω_1 with 0 as weakest condition, then $\mathcal{P}_{\alpha+1}$ forces that $\dot{G}_\alpha^+ \neq \{0\}$ and therefore, generates a $V^{\mathcal{P}_\alpha}$ -generic filter over $\dot{\mathcal{R}}$.*

Proof. See remark 3.5. \square

Lemma 3.8. *(CH) If $\kappa^{\aleph_1} = \kappa$, then \mathcal{P}_κ forces $2^{\aleph_0} = \kappa$.*

Proof. This follows from the fact that there are κ -many ordinals $\alpha < \kappa$ such that $\Phi(\alpha)$ is Cohen forcing (since Cohen forcing satisfies the countable chain condition, and therefore is finitely proper), together with the fact that there are exactly κ -many nice \mathcal{P}_κ -names for subsets of ω (see Lemma 3.2). \square

We are ready to prove our main theorem.

Proof of Theorem 1.3. Our forcing will of course be \mathcal{P}_κ . By Lemmas 3.2, 3.4 and 3.8, it suffices to show that \mathcal{P}_κ forces $\text{PFA}^*(\omega_1)$. But this follows easily from the following claim together with Lemma 3.7.

Claim. *If $\dot{\mathcal{R}}$ is a \mathcal{P}_κ -name for a finitely proper poset and $(\dot{D}_i)_{i < \omega_1}$ is a sequence of \mathcal{P}_κ -names for dense subsets of $\dot{\mathcal{R}}$, then there is a high enough $\alpha < \kappa$ such that $\dot{\mathcal{R}}$ and $(\dot{D}_i)_{i < \xi}$ appear in $V^{\mathcal{P}_\alpha}$ and $\Phi(\alpha) = \dot{\mathcal{R}}$ is a \mathcal{P}_α -name for a V -finitely proper forcing (with respect to $V^{\mathcal{P}_\alpha}$).*

Proof. By the \aleph_2 -chain condition of \mathcal{P}_κ , together with Lemma 3.1 and with the fact that the relevant information about $\dot{\mathcal{R}}$ and $(\dot{D}_i)_{i < \omega_1}$ is decided by a collection of \aleph_1 -many maximal antichains of \mathcal{P}_κ , there is some $\alpha < \kappa$ such that $\{\dot{\mathcal{R}}, (\dot{D}_i)_{i < \omega_1}\} \in V^{\mathcal{P}_\alpha}$. Furthermore, by the choice of Φ we may assume $\Phi(\alpha) = \dot{\mathcal{R}}$. Therefore we will be done if we show that $\dot{\mathcal{R}}$ is V -finitely proper with respect to $V^{\mathcal{P}_\alpha}$. The witnessing club for this will be the set D of all $N^* \cap H(\kappa)^V$ such that $N^* \in \mathcal{M}_\kappa^*$ and $\alpha \in N^*$. D is clearly a club in V .

Now let $q \in \mathcal{P}_\alpha$, $q = (p, \Delta_q)$, let $\{N_i : i < m\} \subseteq D$ be a finite set such that $\{N_i : i < m\} \subseteq \Delta_q$, and let $\nu \in \bigcap_i N_i \cap \omega_1$. Then $q^* := (p, \Delta_q \cup \{(N_i, \text{sup}(N_i \cap \kappa)) : i < m\})$ is clearly a condition in \mathcal{P}_κ extending q . Let us work in $V^{\mathcal{P}|q^*}$. For each i , since $N_i = N^* \cap H(\kappa)$ for some $N^* \in \mathcal{M}_\kappa^*$ and q^* is \mathcal{P}_κ -pre-generic for N_i , we have that $N_i[\dot{G}_\kappa] \cap V = N_i$ by Lemma 3.4. By finite properness of $\dot{\mathcal{R}}$ there is then some condition $\nu^* <_{\omega_1} \dot{\mathcal{R}}$ -extending ν and $(N_i[\dot{G}_\kappa], \dot{\mathcal{R}})$ -generic for all i . But since, for all $i < m$, $N_i[\dot{G}_\kappa] \cap \omega_1 = N_i \cap \omega_1 = N_i[\dot{G}_\alpha] \cap \omega_1$, it follows that ν^* is also $(N_i[\dot{G}_\alpha], \dot{\mathcal{R}})$ -generic for all such i . This finishes the proof since then q can be extended to a \mathcal{P}_α -condition forcing that ν^* is $(N_i[\dot{G}_\alpha], \dot{\mathcal{R}})$ -generic for all $i < m$. \square

4. APPLICATIONS

We are going to show two consequences of the forcing axiom $\text{PFA}^*(\omega_1)$.

4.1. Code(even-odd). In this subsection we consider a principle defined by T. Miyamoto in [4]. This principle follows from BPFA and implies both the negation of Weak Club Guessing and $2^{\aleph_0} = 2^{\aleph_1}$.

From now on, $\text{Lim}(\omega_1)$ will denote the set of all limit ordinals below ω_1 .

Definition 4.1 (Miyamoto). *Code(even-odd):* For every ladder system $\mathcal{A} = \langle A_\delta : \delta \in \text{Lim}(\omega_1) \rangle$ and every $B \subseteq \omega_1$ there exist two clubs C and D of ω_1 such that for each $\delta \in D$: If $\delta \in B$ (resp. $\delta \notin B$), then $|A_\delta \cap C| < \aleph_0$ is odd (resp. even).

We will now prove the following result.

Proposition 4.2. $\text{PFA}^*(\omega_1)$ implies *Code(even-odd)*.

In the following, it will be convenient to denote by $LIND$ the set of all those ordinals which are a limit of indecomposables.¹⁷

We will prove that each instance of $Code(\text{even-odd})$ follows from $PFA^*(\omega_1)$. So, let $\mathcal{A} = \langle A_\delta : \delta \in Lim(\omega_1) \rangle$ be a ladder system and B a subset of ω_1 . We should prove the existence of two clubs C and D of ω_1 such that for every $\delta \in D$,

- (1) If $\delta \in B$, then $|A_\delta \cap C| < \aleph_0$ is odd.
- (2) If $\delta \notin B$, then $|A_\delta \cap C| < \aleph_0$ is even.

For so doing, let us consider the notion of forcing $\mathbb{P}_{\mathcal{A}}$ defined as follows. Its elements are pairs $(f, \langle b_\delta : \delta \in D \rangle)$ such that:

- (a) There exists a normal function $F : \omega_1 \rightarrow \omega_1$ such that f is a finite subset of F .
- (b) Letting $C = \text{range}(f)$, D is included in the set of all ordinals in $C \cap LIND$ which are fixed points of f .
- (c) For each $\delta \in D$, b_δ is finite and $C \cap A_\delta = b_\delta$. Further, if $\delta \in B$ ($\delta \notin B$), then $|b_\delta|$ is odd (even).
- (d) There exists a normal function $F : \omega_1 \rightarrow \omega_1$ extending f and such that: For every $\delta' \in D$ and every $\delta \in C$ with $\delta < \delta'$, F omits all points of $(A_{\delta'} \cap (\delta + 1)) \setminus b_{\delta'}$. That is, if $\gamma \in (A_{\delta'} \cap (\delta + 1)) \setminus b_{\delta'}$, then there exist π , β and β' such that $\beta < \gamma < \beta'$ and $(\pi, \beta), (\pi + 1, \beta') \in F$.

Given $\mathbb{P}_{\mathcal{A}}$ -conditions $(f^\epsilon, \langle b_\delta^\epsilon : \delta \in D^\epsilon \rangle)$ for $\epsilon \in \{0, 1\}$, p_1 extends p_0 iff

- (i) $f_0 \subseteq f_1$,
- (ii) $D_0 \subseteq D_1$, and
- (iii) $b_\delta^1 = b_\delta^0$ for all $\delta \in D_0$.

Lemma 4.3. *Let $p = (f, \langle b_\delta : \delta \in D \rangle) \in \mathbb{P}_{\mathcal{A}}$. Then, for every $\beta < \omega_1$ there is a condition $p' \in \mathbb{P}_{\mathcal{A}}$ extending p and such that $\beta \in \text{dom}(f^{p'})$.*

Proof. Fix a normal function $F : \omega_1 \rightarrow \omega_1$ such that $f \subseteq F$. Let us start with the case when $\delta_\beta = \min(D \setminus \beta)$ exists. This case has 2 subcases.

Subcase 1.1: $\text{dom}(f) \cap [\beta, \delta_\beta) = \emptyset$. We may assume $C \cap \delta_\beta \neq \emptyset$. Let $\delta'_\beta < \delta_\beta$ be an indecomposable ordinal above both β and $\mu = \max(C \cap \delta_\beta)$ (recall that δ_β is a limit of indecomposables). Let

$$\eta = \max(\{\beta\} \cup \bigcup \{A_\delta \cap \delta'_\beta : \delta \in D \setminus \delta'_\beta\})$$

¹⁷A countable ordinal δ is said to be indecomposable if and only for every $\beta < \delta$, the order type of $\delta \setminus \beta$ is δ .

Let τ be such that $f(\tau) = \mu$, and let ε be the unique ordinal such that $\tau + 1 + \varepsilon = \beta$. Finally, let

$$f' = f \cup \{(\tau + 1, \eta + 1), (\beta, \eta + 1 + \varepsilon)\}$$

Since $\eta + 1 + \varepsilon < \delta'_\beta$, it is clear that the result of replacing f with f' in p is a condition p' as required.

Subcase 1.2: $\text{dom}(f) \cap [\beta, \delta_\beta) \neq \emptyset$. Since $(C \cap \delta_\beta) \setminus (\beta + 1)$ is nonempty, adding $(\beta, F(\beta))$ to f does not interfere with any A_δ (for $\delta \in D \setminus \beta$). Hence, the result of replacing f with $f' = f \cup \{(\beta, F(\beta))\}$ in p is a condition as required.

The case when there is no $\delta \in D$ such that $\delta > \beta$ is similar to Subcase 1.2. \square

The following lemma can be proved easily using Lemma 4.3.

Lemma 4.4. *Let $p = (f, \langle b_\delta : \delta \in D \rangle)$ be a condition in \mathbb{P}_A . Then, for every $\beta < \omega_1$ there exists a condition $p' \in \mathbb{P}_A$ extending p and such that, if β is not in the range of $f^{p'}$, then there exists an ordinal $\varepsilon < \beta$ such that $f^{p'}(\varepsilon) < \beta < f^{p'}(\varepsilon + 1)$.*

Lemma 4.5. *Let $p = (f, \langle b_\delta : \delta \in D \rangle)$ be a condition in \mathbb{P}_A . Then, for every fixed point δ of f , $\delta \in \text{LIND}$, there exists a condition $p' \in \mathbb{P}_A$ extending p and such that $\delta \in D^{p'}$.*

Proof. Suppose $\delta \notin D$. We may assume that $\bar{\varepsilon} = \max(C \cap \delta)$ exists. We first extend f to a function \tilde{f} by adding a finite number of pairs (ξ, γ) with $\gamma < \bar{\varepsilon}$ to f in such a way that

- (i) \tilde{f} can be extended to a normal function,
- (ii) $\text{range}(\tilde{f}) \cap A_\epsilon = C \cap A_\epsilon$ whenever $\epsilon \in D$, and
- (iii) for every $\gamma \in (A_\delta \cap (\bar{\varepsilon} + 1)) \setminus \text{range}(\tilde{f})$ there are π and $\beta < \gamma < \beta'$ such that $(\pi, \beta), (\pi + 1, \beta') \in \tilde{f}$.

This is easy to arrange since all sets A_ϵ have order type ω and all relevant ϵ 's are limits of indecomposable ordinals. Now we may extend \tilde{f} to f' if necessary by adding one further pair (ξ, γ) to \tilde{f} in such a way that $\text{range}(f') \cap A_\delta$ has even or odd size according to whether $\delta \in B$ or $\delta \notin B$. Moreover we can make sure that γ is the least member of A_δ above $\bar{\varepsilon}$. This guarantees that all points in $A_\delta \setminus \text{range}(f')$ below $\max(\text{range}(f' \upharpoonright \delta)) + 1$ are omitted by f' and that $(\text{range}(f') \cap A_\epsilon) = C \cap A_\epsilon$ for every $\epsilon \in D$. For $\epsilon \leq \bar{\varepsilon}$ this is obvious, for $\epsilon > \delta$ this is by condition (d) for p .

Finally we extend p by replacing f with f' and by adding the pair $\langle \delta, A_\delta \cap f' \rangle$ to its \bar{b} -part. \square

Corollary 4.6. *If G is \mathbb{P}_A -generic and $F = \bigcup \{f : (\exists \vec{b})(f, \vec{b}) \in G\}$, then $F : \omega_1 \rightarrow \omega_1$ is a normal function. Let $C = \text{range}(F)$ and let D be the set of all fixed points of F in $LIND$. Then C and D are both clubs of ω_1 and for each $\delta \in D$ the following holds.*

- (1) *If $\delta \in B$, then $A_\delta \cap C$ is a finite set of size an odd integer.*
- (2) *If $\delta \notin B$, then $A_\delta \cap C$ is a finite set of size an even integer.*

Obviously, \mathbb{P}_A has cardinality ω_1 . It is also easy to check that if $\{N_i : i \in m\}$ is a finite set of countable elementary substructures of $H(\omega_2)$ containing \mathbb{P}_A and $p = (f, \langle b_\delta : \delta \in D \rangle)$ is an element of \mathbb{P}_A such that each δ_{N_i} is a fixed point of f and $\delta_{N_i} \in D$, then p is (N_i, \mathbb{P}_A) -generic for all i . Proposition 4.2 follows immediately from this together with Corollary 4.6.

Doing minor modifications in the definition of the above forcing it is easy to derive many similar statements from $\text{PFA}^*(\omega_1)$. For example one can check that the negation of *Very Weak Club Guessing* (VWCG) follows from $\text{PFA}^*(\omega_1)$, where VWCG is the assertion that there is a collection \mathcal{A} of size \aleph_1 consisting of subsets of ω_1 of order type ω such that every club of ω_1 has infinite intersection with some $A \in \mathcal{A}$.¹⁸

4.2. $\neg\mathcal{U}$. Our second application is the following.

Proposition 4.7. *$\text{PFA}^*(\omega_1)$ implies the failure of \mathcal{U} .*

Before we start the proof of Proposition 4.7, it will be convenient to introduce the following natural notion of rank of an ordinal with respect to a set of ordinals.

Definition 4.8. Given a set X and an ordinal δ , we define the *Cantor–Bendixson rank of δ with respect to X* , $\text{rank}(X, \delta)$, by specifying that

- (•) $\text{rank}(X, \delta) \geq 1$ if and only if δ is a limit point of ordinals in X .
- (•) If $\mu > 1$, $\text{rank}(X, \delta) \geq \mu$ if and only if for every $\eta < \mu$, δ is a limit of ordinals ϵ with $\text{rank}(X, \epsilon) \geq \eta$.

We will prove that each instance of $\neg\mathcal{U}$ follows from $\text{PFA}^*(\omega_1)$. So, let $\mathcal{G} = \langle g_\delta : \delta \in \omega_1 \rangle$ be such that each g_δ is a continuous function from δ into ω with respect to the order topology. Let $\mathbb{P}_\mathcal{G}$ be the forcing notion consisting of all pairs $(f, \langle k_\xi : \xi \in D \rangle)$ satisfying the following conditions.

- (1) $f \subseteq \omega_1 \times \omega_1$ is a finite strictly increasing function.
- (2) For every $\xi \in \text{dom}(f)$, $\text{rank}(f(\xi), f(\xi)) \geq \xi$.
- (3) $D \subseteq \text{dom}(f)$ and for every $\xi \in D$,

¹⁸In other words, VWCG says the same thing as WCG but allowing \aleph_1 -many cofinal subsets of δ for each $\delta \in \text{Lim}(\omega_1)$.

- (3.1) $k_\xi < \omega$,
- (3.2) $g_{f(\xi)} \text{“range}(f) \subseteq \omega \setminus \{k_\xi\}$, and
- (3.3) $\text{rank}(\{\gamma < f(\xi) : g_{f(\xi)}(\gamma) \neq k_\xi\}, f(\xi)) = \text{rank}(f(\xi), f(\xi))$.

Given conditions $p_\epsilon = (f_\epsilon, \langle k_\xi^\epsilon : \xi \in D_\epsilon \rangle) \in \mathbb{P}_G$ for $\epsilon \in \{0, 1\}$, we say that p_1 extends p_0 iff $f_0 \subseteq f_1$, $D_0 \subseteq D_1$, and $k_\xi^1 = k_\xi^0$ for all $\xi \in D_0$.

The following fact is an easy consequence of our definition of rank.

Lemma 4.9. *Given any finite strictly increasing function $f \subseteq \omega_1 \times \omega_1$, if $\text{rank}(f(\xi), f(\xi)) \geq \xi$ for every $\xi \in \text{dom}(f)$, then f can be extended to a normal function $F : \omega_1 \rightarrow \omega_1$.*

Proof. It suffices to prove, for all $\xi < \omega_1$, that if $g \subseteq \omega_1 \times \omega_1$ is a finite function with $\xi \in \text{dom}(g)$, with $\text{rank}(g(\xi), g(\xi)) \geq \xi$ for every $\xi \in \text{dom}(g)$, and such that $\xi_0 = \max(\text{dom}(g \upharpoonright \xi))$ exists, then there is a strictly increasing and continuous function $h : [\xi_0, \xi] \rightarrow [g(\xi_0), g(\xi)]$ with $h(\xi_0) = g(\xi_0)$ and $h(\xi) = g(\xi)$. But the proof of this fact is immediate by induction on ξ and using the definition of rank. \square

Lemma 4.10 and 4.11 are easy to prove by appealing to condition (2) in the definition of \mathcal{P}_G , together with the openness of all $g_\delta^{-1}(n)$. We give only the proof of Lemma 4.11.

Lemma 4.10. *Every \mathcal{P}_G -condition can be extended to a condition $(f, \langle k_\xi : \xi \in D \rangle)$ such that for every normal function $F : \omega_1 \rightarrow \omega_1$ extending f and every $\xi \in D$, if $\xi' = \max(\text{dom}(f \upharpoonright \xi))$ exists, then $g_{f(\xi)} \text{“range}(F \upharpoonright \xi') \subseteq \omega \setminus \{k_\xi\}$.*

Lemma 4.11. *For every $p = (f, \langle k_\xi : \xi \in D \rangle) \in \mathbb{P}_G$ and every $\xi_0 < \omega_1$ there is a condition $p' \in \mathbb{P}_G$ extending p and such that $\xi_0 \in \text{dom}(f^{p'})$. Also, if $\xi \in \text{dom}(f)$ is a limit ordinal and $\epsilon < f(\xi)$, then there is a condition $p' \in \mathbb{P}_G$ and some $\xi' < \xi$ in $\text{dom}(f^{p'})$ such that $f^{p'}(\xi') > \epsilon$.*

Proof. Let us prove the first claim (the second claim is proved similarly). We may assume that $\xi_0 \notin \text{dom}(f)$ and that $\xi_1 = \min(D \setminus \xi_0)$ exists (otherwise the proof is easier).

Note that for every $\xi' > \xi_1$ in D there is some $l_{\xi'} < \omega$, $l_{\xi'} \neq k_{\xi'}$, such that $g_{f(\xi')}(f(\xi_1)) = l_{\xi'}$. Since all $g_{f(\xi')}^{-1}(\{l_{\xi'}\})$ are open in the order topology, we may fix $\eta < f(\xi_1)$ such that $g_{f(\xi')} \text{“} [\eta, f(\xi_1)) = \{l_{\xi'}\}$ for every $\xi' > \xi_1$ in D . Let $X = \{\gamma < f(\xi_1) : g_{f(\xi_1)}(\gamma) \neq k_{\xi_1}\}$.

Since $\text{rank}(X, f(\xi_1)) = \text{rank}(X \setminus \eta, f(\xi_1)) = \text{rank}(f(\xi_1), f(\xi_1)) \geq \xi_1$, we may find $\gamma \in [\eta, f(\xi_1))$ such that $g_{f(\xi_1)}(\gamma) \neq k_{\xi_1}$ and such that $\text{rank}(\gamma, \gamma) \geq \xi_0$.

Now it is easy to check that $p' = (f \cup \{(\xi_0, \gamma)\}, \langle k_\xi : \xi \in D \rangle)$ is a condition extending p . \square

The following lemma is also easy.

Lemma 4.12. *For every $p = (f, (k_\xi : \xi \in D)) \in \mathbb{P}_G$ and every $\xi \in \text{dom}(f)$ there is a condition $p' \in \mathbb{P}_G$ extending p and such that $\xi \in D^{p'}$.*

Corollary 4.13. *If G is \mathbb{P}_G -generic and*

$$C = \text{range}(\bigcup \{f : (\exists \vec{k})(\langle f, \vec{k} \rangle \in G)\}),$$

then C is a club of ω_1^V and for every $\delta \in C$ there is $k_\delta \in \omega$ such that $g_\delta \restriction C \subseteq \omega \setminus \{k_\delta\}$.

Obviously, \mathbb{P}_G has cardinality ω_1 . On the other hand, it is easy to check that if $\{N_i : i \in m\}$ is a finite set of countable elementary substructures of $H(\omega_2)$ containing \mathbb{P}_G and $p = (f, (k_\xi : \xi \in D))$ is an element of \mathbb{P}_G such that for each i :

- (a) δ_{N_i} is a fixed point of f ,
- (b) $\delta_{N_i} \in D$, and
- (c) $\{\beta < \delta_{N_i} : g_{\delta_{N_i}}(\beta) \neq k_{\delta_{N_i}}\}$ is N_j -stationary¹⁹ for every $j \in m$ such that $\delta_{N_i} = \delta_{N_j}$,²⁰

then p is (N_i, \mathbb{P}_A) -generic for all i . Finally notice that if $p = (f, (k_\xi : \xi \in D))$ satisfies (a), it is easy to find a \mathbb{P}_G -extension p' satisfying also (b) and (c). For so doing, let $(\delta_j)_{j < n}$ be the increasing enumeration of $\{\delta_{N_i} : i < m\}$ and suppose that $\{N_i : \delta_{N_i} = \delta_0\} = \{N_{i_0}, \dots, N_{i_{n_0-1}}\}$. Let $\{k_0, \dots, k_{n_0}\}$ be $n_0 + 1$ colours not touched by $g_{\delta_0} \restriction \text{range}(f)$, and note that there exists $k^0 \in \{k_0, \dots, k_{n_0}\}$ such that $\{\beta < \delta_{N_0} : g_{\delta_{N_0}}(\beta) \neq k^0\}$ is N_{i_j} -stationary for every $j \in n_0$. Hence we may make the promise to avoid the colour k^0 in the colouring g_{δ_0} . Now we continue with δ_1 , and by a similar argument we get a colour k^1 we may avoid in the colouring g_{δ_1} . And so on.

This and the above corollary finish the proof of proposition 4.7.

¹⁹The concept of M -stationarity appears in [5].

²⁰Of course, $\delta_{N_i} = \delta_{N_j}$ implies $k_{\delta_{N_i}} = k_{\delta_{N_j}}$.

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